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Tchebycheff Subspaces of Periodic-Type Tchebycheff Spaces

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Let Λ be an $(m + 1)$-dimensional Tchebycheff space of periodic-type functions. Then Λ contains an $(m - 1)$-dimensional Tchebycheff subspace. Consequently, Λ has a Markov-type property: it contains a chain of Tchebycheff spaces $\Lambda = \Lambda_m \supset \Lambda_{m-2} \supset \ldots$, with $\dim \Lambda_i = i + 1$.

1. *T*-SPACES OF PERIODIC TYPE

Let $u_0, u_1, \ldots, u_m$ be defined and continuous on the interval $[a, b]$, forming a Tchebycheff-system (abbr. T-system) on $[a, b)$ and such that $u_i(b) = \alpha u_i(a)$ for some $\alpha \neq 0$ and $i = 0, 1, \ldots, m$ $(m \geqslant 2)$. We call such functions functions of periodic type. In particular, they are periodic or antiperiodic when $\alpha = +1$ or $\alpha = -1$, respectively. Since the determinant

$$U\begin{pmatrix} u_0, u_1, \ldots, u_m \\ t_0, t_1, \ldots, t_m \end{pmatrix} = \begin{vmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_m) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_m) \\ \vdots & \vdots & & \vdots \\ u_m(t_0) & u_m(t_1) & \cdots & u_m(t_m) \end{vmatrix}$$

is positive whenever $a \leqslant t_0 < t_1 < \cdots < t_m < b$ and is continuous in t_m, and the functions involved are of periodic type, it is clear that $\{u_i\}_{i=0}^m$ forms a T-system on $(a, b]$ and hence $(-1)^m \alpha > 0$.

We say that x is a nodal zero of f, in the interval (a, b), if $f(x) = 0$ and if $f(x_1) f(x_2) < 0$ whenever $x - \epsilon < x_1 < x < x_2 < x + \epsilon$ for some $\epsilon > 0$. If $f \in \Lambda_m = \operatorname{span}\{u_0, u_1, \ldots, u_m\}$ on $[a, b)$, we say that a is a nodal zero of f, if $f(a) = 0$ and if $\alpha f(x_1) f(x_2) < 0$ whenever $a < x_1 < a + \epsilon$ and $b - \epsilon < x_2 < b$ for some $\epsilon > 0$. All other zeros of f are called nonnodal zeros.

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Let $f \in \Lambda_m$. We extend f to $[2a - b, 2b - a)$ by defining

$$\begin{aligned} \bar{f}(t) &= \frac{1}{\alpha} f(t + b - a), && 2a - b \leqslant t < a, \\ &= f(t) && a \leqslant t < b, \\ &= \alpha f(t - b + a), && b \leqslant t < 2b - a, \end{aligned}$$

Clearly, the extended functions form a T-space (the span of a T-system) on every half-open interval of length $b - a$, contained in $[2a - b, 2b - a)$.

LEMMA 1. *Let $f \in \Lambda_m$, $f \neq 0$. The number of the zeros of f (in $[a, b)$), $Z(f; [a, b))$ is even or odd, according to m being even or odd (nonnodal zeros counted twice).*

Proof. The claim of the lemma is clear if $f(a) \neq 0$. If $f(a) = 0$, then $\bar{f}(a - \epsilon) \neq 0$ for some $\epsilon > 0$ and $Z(f; [a, b)) = Z(\bar{f}; [a, b)) = Z(\bar{f}; [a - \epsilon, b - \epsilon))$.

2. AN INTERPOLATION PROBLEM

We need the:

DEFINITION [2]. Let f be defined on a real set M and let $t_1, t_2, \ldots, t_k \in M$ with $t_1 < t_2 < \cdots < t_k$.

(i) $t_1, t_2, \ldots, t_k$ form an alternation of f, of length k, if $\operatorname{sgn} f(t_i) = -\operatorname{sgn}(f(t_{i+1})) \neq 0$, for $i = 1, 2, \ldots, k - 1$.

(ii) $t_1, t_2, \ldots, t_k$ form a quasialternation of f, of length k if $\operatorname{sgn}(f(t_i) - f(t_{i+1})) = -\operatorname{sgn}(f(t_{i+1}) - f(t_{i+2}))$, for $i = 1, 2, \ldots, k - 2$.

We also need the following results of Zielke [2, Lemmas 3, 5 and 7]:

Let $v_1 = 1, v_2, \ldots, v_{m+1}$ be continuous functions on (a, b), forming a CT-system (Complete Tchebycheff-system) (i.e., $v_1, v_2, \ldots, v_r$ form a T-system for $r = 1, 2, \ldots, m + 1$) on this interval. v_2 is strictly increasing and hence $I = v_2((a, b))$ is an open interval and $f_1, f_2, \ldots, f_{m+1}$, defined by $f_i = v_i \circ v_2^{-1}$, $i = 1, 2, \ldots, m + 1$ form a CT-system on I. We have that $g_2 = Df_2 = 1$, $g_3 = Df_3, \ldots, g_{m+1} = Df_{m+1}$ is a CT-system on I, D being the right-hand side derivative.

Let $J = g_3(I)$, $h_i = g_i \circ g_3^{-1}$ and $J^* = (\inf J, \sup J)$. We extend the functions h_i, $i = 2, 3, \ldots, m + 1$, to $\bar{h}_i$ on J^* [2, Lemma 5]. Every $\bar{h} \in \operatorname{span}\{\bar{h}_2, \bar{h}_3, \ldots, \bar{h}_{m+1}\}$ has no alternation of length $>m$ and since every $h \in \operatorname{span}\{h_2, h_3, \ldots, h_{m+1}\}$ has no quasialternation of length $>m$ and the extension of the h's is by linear segments, the same is true for the $\bar{h}$'s. Hence $D\bar{h}$ has no alternation of length $>m - 1$.

Finally, we define

$$D_1 v = D(v \circ v_2^{-1})$$

and

$$D_2 v = D\bar{h},$$

where $h = g \circ g^{-1}$ and $g = D_1 v$.

Let $\Lambda_m = \text{span}\{u_0, u_1, ..., u_m\}$, $m \geqslant 2$, be a T-space on $[a, b)$ with $u_i(b) = \alpha u_i(a)$ for some $\alpha \neq 0$ and $i = 0, 1, ..., m$. We may assume that $u_i(a) = 0$ for $i = 1, 2, ..., m$ and that $\{u_1, u_2, ..., u_m, u_0\}$ is a CT-system on (a, b). Define

$$v_i = \frac{u_i \mid_{(a,b)}}{u_1 \mid_{(a,b)}}, \qquad \text{for } i = 1, 2, ..., m$$

and

$$v_{m+1} = \frac{u_0 \mid_{(a,b)}}{u_1 \mid_{(a,b)}}$$

LEMMA 2. *Let $v_1, v_2, ..., v_{m+1}$ be defined as above and let $x, t_1, t_2, ..., t_{m-1}$ be m points in the interval (a, b) with $t_1 < t_2 < \cdots < t_{m-1}$. The trivial polynomial (in the v_i's) is the only element of* $\text{span}\{v_1, v_2, ..., v_{m+1}\}$ *satisfying*

(1) $v(t_i) = 0$, $i = 1, 2, ..., m - 1$,
(2) $D_1 v(x) = 0$, *and*
(3) $D_2 v(x) = 0$.

Proof. Assume, to the contrary, that there exists a non-trivial polynomial, $v = \sum_{i=1}^{m+1} a_i v_i$ that satisfies (1), (2) and (3). Consider the polynomial $u = \sum_{i=0}^{m} a_i u_i$ (with $a_0 = a_{m+1}$). By Lemma 1, u vanishes at an additional point t_0 in $[a, b)$, or, exactly one of its zeros is a nonodal one.

Assume, first, that $u(a) \neq 0$. We can split the interval $v_2((a, b))$ into m consecutive subintervals on which $f = v \circ v_2^{-1} = \sum_{i=1}^{m+1} a_i f_i$, is increasing and decreasing alternately, thus $D_1 v$ has an alternation of length m. By [2, Lemma 3] $D_1 v$ has no quasialternation of length $>m$ and if $D_1 v$ is not identically zero (in which case $v = 0$), $D_2 v$ has an alternation of length $m - 1$. Since $D\bar{h}$ changes sign at points where $\bar{h}$ is negative or positive, $D_2 v$ does not change sign at x. Let $s_1, s_2, ..., s_{m-2}$ be points in J^* such that in each of the intervals $(s_0, s_1), (s_1, s_2), ..., (s_{m-2}, s_{m-1})$ ($s_0 = \inf J$ and $s_{m-1} = \sup J$), $D\bar{h}$ is always nonnegative or nonpositive. $y = (g_3 \circ v_2)(x)$ belongs to one of them, say (s_j, s_{j+1}), and assume for the sake of simplicity that $D\bar{h}$ is nonnegative on this interval. Clearly, there exist two points $s_j < y_1 < y < y_2 < s_{j+1}$, such that $D\bar{h}(y_1) > 0$ and $D\bar{h}(y_2) > 0$ and since $D\bar{h}(x) = 0$, $D\bar{h}$ has a quasialternation of length $\geqslant m + 1$. Hence $v = 0$.

If $u(a) = 0$, then $a_0 = 0$ $(a_{m+1} = 0)$ and the assertion holds in span $\{v_1|_{(a,b)}, \ldots, v_m|_{(a,b)}\}$.

3. The Main Result

It is known that if Λ is an n-dimensional T-space on a set M, then Λ contains an $(n-1)$-dimensional T-space iff we can extend the elements of Λ to $M \cup \{x\}$, $x \notin M$ and the extended functions form a T-space on that set. This is not the case here. However, we show that we can, by adding a "double point" to $[a, b)$, construct an $(n-2)$-dimensional T-subspace.

THEOREM 1. *Let $u_0, u_1, \ldots, u_m$ be defined and continuous on the interval $[a, b]$, forming a* T-*system on $[a, b)$, and such that $u_i(b) = \alpha u_i(a)$ for some $\alpha \neq 0$ and $i = 0, 1, \ldots, m$. The T-space,* span $\{u_0, u_1, \ldots, u_m\}$ *contains an $(m-1)$-dimensional T-subspace, on $[a, b)$.*

Proof. For $m = 2$, span $\{u_0, u_1, u_2\}$ contains a positive function on $[a, b)$ so we assume $m \geqslant 3$.

We may assume that $u_i(a) = 0$ for $i = 1, 2, \ldots, m$ and that

$$\{u_1|_{(a,b)}, u_2|_{(a,b)}, \ldots, u_m|_{(a,b)}, u_0|_{(a,b)}\} \qquad \text{is a CT-system.}$$

Define v_i, $i = 1, 2, \ldots, m+1$ as in Lemma 2, and since $D_1 v_1 = D_2 v_2 = D_2 v_1 = 0$ $D_1 v_2 = 1$ and $D_2 v_3 = 1$, we can define $\tilde{v}_1 = v_1$ and for $i \geqslant 4$, $\tilde{v}_1 = v_i - a_i v_2 - b_i v_3$ such that $D_1 \tilde{v}_i(x) = D_2 \tilde{v}_i(x) = 0$ for a given x, $a < x < b$.

The determinant of the interpolation problem of Lemma 2 does not vanish and hence

$$U\begin{pmatrix} \tilde{v}_1, \tilde{v}_4, \ldots, \tilde{v}_{m+1} \\ t_1, t_2, \ldots, t_{m-1} \end{pmatrix} \neq 0$$

whenever $a < t_1 < t_2 < \cdots < t_{m-1} < b$, which implies

$$U\begin{pmatrix} \tilde{u}_1, \tilde{u}_4, \ldots, \tilde{u}_m, \tilde{u}_0 \\ t_1, t_2, \ldots, t_{m-2}, t_{m-1} \end{pmatrix} \neq 0,$$

$\tilde{u}_1 = u_1$ and $\tilde{u}_i = u_i - a_i u_2 - b_i u_3$, $i = 0, 4, 5, \ldots, m$.

$\tilde{u}_1|_{((a,b)}, \tilde{u}_4|_{(a,b)}, \ldots, \tilde{u}_m|_{(a,b)}$ and $\tilde{u}_0|_{(a,b)}$ form a T-system on (a, b). We can extend these functions to $[a, b)$ and the extended functions will form T-system on this interval [1]. This extension is continuous, as will be shown in Theorem 3 and hence, span $\{\tilde{u}_0, \tilde{u}_1, \tilde{u}_4, \ldots, \tilde{u}_m\}$ is a T-space on $[a, b)$.

THEOREM 2. *Let $u_0, u_1, \ldots, u_m$ be as in Theorem 1 and $\Lambda_m =$*

span $\{u_0, u_1, \ldots, u_m\}$. *There exists a chain of T-spaces*: $\Lambda_m \supset \Lambda_{m-2} \supset \cdots \supset \Lambda_{i_0}$ ($i_0 = 0, 1$, *according to m being even or odd*) *with* $\dim \Lambda_i = i + 1$.

THEOREM 3. *Let* $u_0, u_1, \ldots, u_m$ *be continuous functions on* $[a, b)$, *forming a T-system on this interval. If* $\Lambda_m = \text{span}\{u_0, u_1, \ldots, u_m\}$ *does not contain an m-dimensional T-subspace then it contains an* $(m - 1)$*-dimensional one.*

Proof. As in [1] we define $w(t) = \max_{0 \leqslant i \leqslant m} |u_i(t)|$, $a \leqslant t < b$, *and* $v_i(t) = u_i(t)/w(t)$, $i = 0, 1, 2, \ldots, m$; $a \leqslant t < b$. It is sufficient to prove the theorem for span $\{v_0, v_1, \ldots, v_m\}$. We choose a sequence $\{t_j\}$ in $[a, b)$ with $\lim_{j\to\infty} t_j = b$. This sequence contains a subsequence $\{t_{j_k}\}$ such that $\lim_{k\to\infty} v_i(t_{j_k})$ exist for $i = 0, \ldots, m$.

We show that the limits do not depend on $\{t_j\}$. Assume that for some i_1 there exist two sequences $\{s_j\}$ and $\{r_j\}$ such that $\lim_{j\to\infty} v_{i_1}(s_j) = S < \lim_{j\to\infty} v_{i_1}(r_j) = R$. Since for some i_0, $v_{i_0} = 1$ near b, $v_{i_1} - ((S + R)/2)\, v_{i_0}$ vanishes infinitely many times in contradiction to the fact that $\{v_i\}_{i=0}^m$ is a T-system. Thus we may extend v_i, $i = 0, 1, \ldots, m$ to $[a, b]$ and the extended functions are continuous on $[a, b]$.

By [1], if Λ_m does not contain an m-dimensional T-subspace, $(v_0(a), v_1(a), \ldots, v_m(a))$ and $v_0(b), v_1(b), \ldots, v_m(b))$ are proportional, and thus by Theorem 1, Λ_m contains an $(m - 1)$-dimensional T-subspace.

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